Quantifier Symbolization Advice
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I offer below various bits of advice about advanced quantifier translation in First-order Logic. They are meant to complement—and indeed cover some points also expressed in—*forall* x.

1. To understand the meaning of a symbolized sentence, first read the parts to yourself literally and *in order*. Read predicates starting with the name or variable. So, first read "$\forall x \neg P(x)$" as "For every x, it's not the case that x is P." Then, as a second step, you can reason that that says "Everything is not P."

2. Focus on the meanings of these four basic sentences, and try to internalize their differences:

   $\forall x \neg P(x)$ = Everything is not P; nothing is P.

   $\neg \forall x P(x)$ = Not everything is P.

   $\exists x \neg P(x)$ = Something is not P.

   $\neg \exists x P(x)$ = Nothing is P.

   The first and fourth of these are logically equivalent. The second and third are also logically equivalent. Think about why. I think of this as "To preserve meaning, if you move the negation in or out, switch the quantifier."

3. Faced with a compound sentence, first think: Is this an (unquantified) truth-functional compound of simpler sentences? If it's a conjunction, disjunction, conditional, or biconditional, you can divide your task into more-manageable portions.

4. Most quantified compound sentences you will encounter are either *universally-quantified conditionals* or *existentially quantified conjunctions*. (Within such sentences, the antecedent, consequent, or conjuncts may themselves involve quantifiers, but don't let that distract you from identifying the *overall* shape of the sentence.) So, which one?

   (a) Does the sentence express a relationship between sets? Does it say "Things like *this* are like *that*" or "Things in this set are in that set" or "Things with these properties also have these further properties"? If so, its overall structure is probably $\forall x[P(x) \rightarrow Q(x)]$

   (b) Does the sentence express that at least one thing, or some things have a property? Does it say that something exists and it's like *this*? If so, its overall structure is probably $\exists x(P(x) \wedge Q(x))$

5. The overall structure of *many* quantified English sentences is one of these four types. They are so common that Aristotle gave them names (A, E, I, and O sentences). To me these names are just one more thing to remember, for no reason. But you *should* recognize their structures. Here they are with sample translations:

   $\forall x(T(x) \rightarrow F(x))$ = All tigers are fierce.

   $\forall x(T(x) \rightarrow \neg F(x))$ = No tigers are fierce.

   $\exists x(T(x) \wedge F(x))$ = Some tigers are fierce.

   $\exists x(T(x) \wedge \neg F(x))$ = Some tigers are not fierce.

6. An **alarm** should go off in your head if you're ever tempted to write a sentence whose structure is existentially-quantified conditional or universally-quantified conjunction. In the first case, you're saying something strange, like $\exists x[F(x) \rightarrow T(x)]$: "Something exists such that if it's fierce, it's a tiger." This sentence is *true* if there are non-fierce chipmunks! That's probably not what you mean. Similarly, $\forall x[T(x) \wedge F(x)]$ says that everything is a fierce tiger. It does not say that all tigers are fierce. There are contexts (say, in philosophical arguments) where you might want to say that everything in the domain is a member of two sets, like "Everything is physical and extended." But the alarm should provoke you to make sure that's really what you want to say. Existentially-quantified conditionals should just be avoided. Remember: **alarm**!

7. Let's put together the ideas in #2 and #5. Using them together, we can see why two common sentence-types are equivalent.

   $\neg\forall x(T(x) \to F(x))$ is logically equivalent to $\exists x(T(x) \land \neg F(x))$

   Why? Up in #2 I suggested you think about why "not all" and "something's not" are equivalent. So, $\neg\forall x(T(x) \to F(x))$ is equivalent to $\exists x \neg(T(x) \to F(x))$. Now think back about what a negated conditional is equivalent to: $\neg(X \to Y) \leftrightarrow (X \land \neg Y)$. (Remember that the one case in which a conditional is false is where its antecedent is true and its consequent is false.) Substitute the logically-equivalent conjunction and you've got the second sentence.

8. Students sometimes express confusion about when to use "every" and "any." Unfortunately, there is not an exact correlation between these terms in English and FOL quantifiers. You need to think about how many things a claim is about. Here are some examples, using people for a domain:

   $$\left.\begin{array}{l}\text{Anyone shorter than Joe is shorter than Mick. } \forall x(S(x,j) \to S(x,m)) \\ \text{Everyone shorter than Joe is shorter than Mick. } \forall x(S(x,j) \to S(x,m)) \\ \text{If anyone is shorter than Joe, they're shorter than Mick. } \forall x(S(x,j) \to S(x,m))\end{array}\right\} \text{same FOL sentences}$$

   $$\left.\begin{array}{l}\text{If anyone's shorter than Joe, Paul is. } \exists x S(x,j) \to S(p,j) \\ \text{If everyone's shorter than Joe, Paul is. } \forall x S(x,j) \to S(p,j)\end{array}\right\} \text{not the same FOL sentences}$$

   $$\left.\begin{array}{l}\text{Joe isn't shorter than everyone. } \neg\forall x S(j,x) \\ \text{Joe isn't shorter than anyone. } \neg\exists x S(j,x)\end{array}\right\} \text{not the same FOL sentences}$$

9. Similarly, *most* occurrences of "some" should be translated with an existential quantifier, like "Some tigers are fierce" and "Someone's shorter than Joe." However, here is an example of a sentence with "some" that's universal:

   If someone's shorter than Joe, they are shorter than Mick. $\forall x(S(x,j) \to S(x,m))$

   Why? Despite the appearance of "someone," notice that this sentence means the same thing as several sentences above about "anyone." How? Notice the reference back to that "someone" in the consequent. This sentence is about the relationship between sets (the set 'shorter than Joe' and the set 'shorter than Mick'). Consider how different the following sentence is:

   If someone's shorter than Joe, someone is shorter than Mick. $\exists x S(x,j) \to \exists x S(x,m)$

10. Relatedly, for any sentence P that does *not* include the variable x (for example, the TFL sentence P, or $S(m,j)$, or $\forall y M(y)$), this equivalence holds:

    $\exists x F(x) \to P$ is logically equivalent to $\forall x(F(x) \to P)$

    If that's confusing, read the sentences out loud in logic-ese. They both say "If anything's F, P," as in "If anything's fierce, I'm getting out of here pronto." The second sentence encloses the consequent in parentheses. So checking whether it's true would involve checking each thing in the domain for whether it fulfills the antecedent. And that is equivalent to checking whether $\exists x F(x)$.

11. Another set of sentences whose meanings you should internalize are the four basic sentences with two overlapping quantifiers:

    $\forall x \forall y L(x,y) =$ Everyone loves everyone.

    $\forall x \exists y L(x,y) =$ Everyone loves someone.

    $\exists x \forall y L(x,y) =$ Someone loves everyone.

    $\exists x \exists y L(x,y) =$ Someone loves someone.

    Notice how the order of the quantifiers matters to the meaning.

12. Then, consider these variants in which the order of variables is reversed:

    $\forall x \exists y L(y, x) =$ Everyone is loved by someone.
    $\exists x \forall y L(y, x) =$ There's someone everyone loves.

    "Everyone is loved by someone" does *not* mean the same thing as "Someone loves everyone." Similarly, "There's someone everyone loves" does *not* mean the same thing as "Everyone loves someone." The order of the variables also matters.

13. Sometimes you need to add a quantifier within a sentence. Here is one very typical example of a compound sentence which it would be profitable for you to think about:

    Some non-fierce tigers are hungrier than every fierce tiger.

    Clearly this sentence is an existential conjunction, because it says "There are some things like this which are also like that." It starts out $\exists x(\neg F(x) \wedge T(x))$ (There are non-fierce tigers). It's going to end up saying something about the relationship between those non-fierce tigers (now x) and something else. It will say they are hungrier than y: $H(x, y)$, where $H(x, y)$ means "x is hungrier than y." That will be the last term. But in the middle we have to say something about what it is that they're hungrier than: every fierce tiger. So we want to use y to represent "all fierce tigers." For that, we can write "everything that's fierce and a tiger" and we should start by thinking about $\forall y(F(y) \wedge T(y))$.

    Wait, you might think: doesn't that say that everything is a fierce tiger? Yes, that sentence does. That's the sort of sentence that raises an alarm! But we're going to use this conjunction of properties as the antecedent of a conditional, to write that if y is both F and T then it's something the non-fierce tigers are hungrier than. So we're going to extend the scope of the universal quantifier to cover the consequent. Since the universal quantifier will now cover a conditional, there's no more problem here than there is in saying $\forall z(T(z) \rightarrow F(z))$, "All tigers are fierce." We're not saying there that everything is a tiger, just that the set of fierce tigers stands in a certain relation (the predicate H) to the set of non-fierce tigers. The result is:

    $$\exists x\{(\neg F(x) \wedge T(x)) \wedge \forall y[(F(y) \wedge T(y)) \rightarrow H(x, y)]\}$$

    Or, in logic-ese: there's something that's not fierce and a tiger, and if anything is fierce and a tiger, then the first thing is hungrier than the second thing. That's equivalent to the more natural sentence above.

14. Students sometimes aren't sure whether to put quantifiers at the beginning of a compound sentence or inside it. Here's a recommendation: Give quantifiers the scopes they need to have, and nothing more. Apply them to the shortest sentences you can. For example, if a variable appears only in the consequent of a conditional, quantify it by making its scope just that consequent. Three reasons: (a) This minimizes the chance of saying something you don't mean; (b) This makes sentences much easier to read; (c) When we get to FOL natural deduction, you will be able to apply the rule $\&E$ (conjunction elimination) to $\exists x T(x) \wedge \exists y T(y)$, because it's a conjunction, but not to $\exists x \exists y(T(x) \wedge T(y))$. In the latter, you have to deal with the quantifiers. It's a double-quantified sentence, not a conjunction.